On t-Motifs

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Chapter 6

Weights and Purity

6.1 Dieudonné t-Modules

6.1.1. Let k and K be as usual. Denote by τ the continuous endomorphism of the field of Laurent series $K((t^{-1}))$ that fixes t^{-1} and that restricts to the q-th power map on K.

Definition. A *Dieudonné t-module*[1] over K is a pair (V, σ) of

- a finite-dimensional $K((t^{-1}))$-vector space V and
- an additive map $\sigma : V \to V$ satisfying $\sigma(fv) = \tau(f)\sigma(v)$ for all $f \in K((t^{-1}))$ and all $v \in V$,

such that $K\sigma(V) = V$.

A morphism of Dieudonné t-modules is of course a $K((t^{-1}))$-linear map commuting with σ.

6.1.2. Dieudonné t-modules are easily classified, at least over a separably closed field. The main 'building blocks' are the following modules:

Definition. Let $\lambda = s/r$ be a rational number with $(r, s) = 1$ and $r > 0$. The Dieudonné t-module V_λ is defined to be the pair (V_λ, σ) with

- $V_\lambda \stackrel{\text{def}}{=} K((t^{-1}))e_1 \oplus \ldots \oplus K((t^{-1}))e_r$

[1] This the equal characteristic analogue of the p-adic object that is commonly called a Dieudonné module.

- $\sigma(e_i) \stackrel{\text{def}}{=} e_{i+1}$ $(i < r)$ and $\sigma(e_r) \stackrel{\text{def}}{=} t^s e_1$

The classification states:

Proposition. *If V is a Dieudonné t-module over a separably closed field K then there exist rational numbers $\lambda_1, \cdots, \lambda_n$ such that*

- $V \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$*, and,*
- *the t^{-1}-adic valuations of the roots of the characteristic polynomial of σ expressed on* any *$K((t^{-1}))$-basis are $\{-\lambda_i\}_i$, each counted with multiplicity* $\dim V_{\lambda_i}$.

If $\lambda \neq \mu$ then $\mathrm{Hom}(V_\lambda, V_\mu) = 0$. For all λ, the ring $\mathrm{End}(V_\lambda)$ is a division algebra over $k((t^{-1}))$. Its Brauer class is $\lambda + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z} = \mathrm{Br}(k((t^{-1})))$.

Note that this classification is formally identical to the classification of the classical (p-adic) Dieudonné modules.[2]

Proof. This is shown in Appendix B of [Laumon 1996]. Although the statements therein are made only for a particular field K, nowhere do the proofs make use of anything stronger then the separably closedness of K. □

6.1.3. The following characterisation of V_λ is useful.

Proposition. *Let V be a Dieudonné t-module over a separably closed field K and λ a rational number. The following are equivalent:*

- $V \approx V_\lambda \oplus V_\lambda \oplus \cdots \oplus V_\lambda$*;*
- *there exists a lattice $\Lambda \subset V$ such that $\sigma^r(\Lambda) = t^s \Lambda$ where r and s are coprime integers with $\lambda = s/r$.*

Proof. One $\Rightarrow$ Two. If $V = V_\lambda$ and (e_i) the basis that occurs in its definition (6.1.2) then the lattice generated by the same basis (e_i) has the required property. For $V = V_\lambda \oplus \cdots \oplus V_\lambda$ it thus suffices to take the lattice $\Lambda \oplus \cdots \oplus \Lambda$.

[2] *See* [Dieudonné 1957].

Two ⇒ One. The operator $t^{-s}\sigma^r$ transforms a $K[[t^{-1}]]$-basis of Λ into a new $K[[t^{-1}]]$-basis of Λ and therefore has eigenvalues of valuation 0. □

6.2 Pure *t*-Motifs

6.2.1. Let K be separably closed. Let M be an effective t-motif over K. Then

$$M((t^{-1})) \stackrel{\text{def}}{=} M \otimes_{K[t]} K((t^{-1})) = M(t) \otimes_{K(t)} K((t^{-1}))$$

is a Dieudonné t-module. The displayed equality shows that it only depends on the isogeny class of M. By the classification of Dieudonné t-modules (6.1.2) there exist rational numbers $\lambda_1, \ldots, \lambda_n$ such that

$$M((t^{-1})) \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}.$$

We call these rational numbers the *weights* of M. If K is not separably closed than we define the weights of an effective t-motif M to be the weights of M_{K^s}. This clearly does not depend on the choice of a separable closure.

We say that M is *pure of weight* λ if the only weight occurring is λ. By Proposition 6.1.3, this coincides with the definition as given in [Anderson 1986].

6.2.2. We now collect a number of facts related to the notions of weights and purity. They are either immediate consequences of the definitions or well-known facts established in the literature.

Proposition. *We have the following:*

- *If M is pure of weight λ then every subquotient of M is pure of weight λ;*
- *If M has a filtration in which all successive quotients are pure of weight λ, then M is pure of weight λ;*
- *If the sets of weights of M_1 and M_2 are disjoint then* $\mathrm{Hom}(M_1, M_2) = 0$*;*
- *Drinfeld modules of rank r are pure of weight $1/r$ (in particular: C is pure of weight 1);*

- *The weights of $M_1 \otimes M_2$ are the sums of weights of M_1 with those of M_2;*
- *The weight of a pure effective t-motif M is non-negative.*

Proofs. One. If M' is a subquotient of M then $M'((t^{-1}))$ is a subquotient of $M((t^{-1}))$ and the claimed statement follows at once from the Classification 6.1.2.

Two. A normal series of M induces a normal series of $M((t^{-1}))$ and again the contention follows from 6.1.2.

Three. $\mathrm{Hom}(M_1, M_2)$ is a submodule of $\mathrm{Hom}(M_1((t^{-1})), M_2((t^{-1})))$, which is zero by 6.1.2.

Four. See Proposition 4.1.1. of [Anderson 1986].

Five. Immediate since the zeroes of the characteristic polynomials are multiplied.

Six. Clear for rank one M, for a general M take the top exterior power. □

6.2.3. If M is an effective t-motif and

$$M((t^{-1})) \approx V_{\lambda_1} \oplus \cdots \oplus V_{\lambda_n}$$

then by the Proposition

$$(M \otimes C)((t^{-1})) \approx V_{\lambda_1+1} \oplus \cdots \oplus V_{\lambda_n+1}.$$

It is thus natural to define the *weights* of a t-motif (M, i) to be the set of $\lambda + i$ where λ runs through the weights of M. To be consistent, a t-motif (M, i) is then said to be *pure of weight* λ if and only if M is pure of weight $\lambda - i$.

6.3 A Digression on Brauer Groups

6.3.1. Let M be a t-motif that is pure of weight λ. Thus $M((t^{-1})) \approx nV_\lambda$ and by Proposition 6.1.2 the endomorphism ring $\mathrm{End}(M((t^{-1})))$ is a central simple algebra whose class in the Brauer group of $k((t^{-1}))$ is $\lambda + \mathbf{Z}$. Thus, the map

$$\{\text{weights}\} = \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} = \mathrm{Br}(k((t^{-1})))$$

has some kind of interpretation in terms of Dieudonné t-modules of pure t-motifs.

6.3.2. Classical motifs also have weights, and these weights form an infinite cyclic group. If we normalise things so that the Lefschetz motif has weight 1,[(3)] then the weight group is $\frac{1}{2}\mathbf{Z}$. A piece of the degree i cohomology $h^i(X)$ then has weight $i/2$. All this seems to harmonise easily with the fact that the Brauer group of $\mathbf{R}$ is cyclic of order two—it suggests that the map

$$\{\text{weights}\} = \tfrac{1}{2}\mathbf{Z} \to \tfrac{1}{2}\mathbf{Z}/\mathbf{Z} = \mathrm{Br}(\mathbf{R}),$$

can be interpreted in a fashion similar to the above.

6.3.3. We shall sketch one possible such interpretation, albeit a somewhat *ad hoc* one. Let X be a smooth and projective variety over $\mathbf{C}$. Put $V = H^i(X(\mathbf{C}), \mathbf{C})$. The complex vector space V comes equipped with a Hodge decomposition

$$V = \oplus_{p+q=i} H^{p,q}.$$

Let α be the anti-linear automorphism of the complexified co-tangent bundle of X that is the composition of the linear automorphism 'multiplication with i' followed by complex conjugation. Then α induces an anti-linear automorphism α_* of V. On the Hodge decomposition it restricts to

$$\alpha_* : H^{p,q} \to H^{q,p} : c \mapsto i^{q-p}\bar{c}. \tag{6.1}$$

The endomorphisms of V that commute with α_* form an $\mathbf{R}$-algebra denoted $\mathrm{End}(V, \alpha_*)$. Starting from (6.1) an easy calculation yields

$$\mathrm{End}(V, \alpha_*) \approx \begin{cases} \mathrm{M}(n, \mathbf{R}) & \text{if } i \text{ even,} \\ \mathrm{M}(\frac{n}{2}, \mathbf{H}) & \text{if } i \text{ odd,} \end{cases}$$

where n stands for the dimension of V and $\mathbf{H}$ for the algebra of Hamilton quaternions. We conclude that the two elements of the Brauer group of $\mathbf{R}$ correspond to the two weight classes modulo $\mathbf{Z}$.

[(3)]This is *not* the customary normalisation—one usually assigns the weight 2 to the Lefschetz motif.